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Parameter Estimation for the Inverted Nadarajah-Haghighi Distribution Based on Median Ranked Set Sampling

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Abstract

The inverted Nadarajah–Haghighi distribution has recently attracted attention due to its flexibility in modeling lifetime data with non-monotonic hazard rate structures. In this study, parameter estimation for this distribution is investigated under two competing sampling designs, namely simple random sampling and median ranked set sampling. Maximum likelihood estimators of the model parameters are derived for both sampling schemes. To evaluate the efficiency of the proposed estimators, an extensive Monte Carlo simulation study is conducted based on bias and mean squared error criteria over different sample sizes and parameter configurations. The simulation results consistently demonstrate that median ranked set sampling provides more precise and efficient parameter estimates compared to simple random sampling. The findings highlight the advantage of incorporating ranking information into the sampling process and show that the proposed approach offers a more effective inference framework for the inverted Nadarajah–Haghighi distribution. This study contributes to the existing literature by presenting a comprehensive comparison of sampling designs for parameter estimation within this distributional setting and by demonstrating the practical gains achieved through median ranked set sampling.

Keywords: Median ranked set sampling, simple random sampling, maximum likelihood estimator, inverted Nadarajah-Haghighi distribution

Medyan Sıralı Küme Örneklemesine Dayalı Ters Nadarajah-Haghighi Dağılımının Parametre Tahmini

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Özet

Ters Nadarajah–Haghighi dağılımı, monoton olmayan tehlike oranı yapısına sahip yaşam süresi verilerinin modellenmesindeki esnekliği nedeniyle son yıllarda dikkat çekmektedir. Bu çalışmada, söz konusu dağılım için parametre tahmini iki rekabetçi örnekleme tasarımı altında incelenmektedir: basit rastgele örnekleme ve medyan sıralı kümeleme örneklemesi. Model parametrelerine ait maksimum olabilirlik tahmin edicileri her iki örnekleme şeması için türetilmiştir. Önerilen tahmin edicilerin etkinliğini değerlendirmek amacıyla, farklı örneklem büyüklükleri ve parametre yapılandırmaları altında yanlılık ve ortalama kare hata ölçütlerine dayalı kapsamlı bir Monte Carlo simülasyon çalışması gerçekleştirilmiştir. Simülasyon sonuçları, medyan sıralı kümeleme örneklemesinin basit rastgele örneklemeye kıyasla daha hassas ve daha etkin parametre tahminleri sağladığını tutarlı biçimde ortaya koymaktadır. Elde edilen bulgular, örnekleme sürecine sıralama bilgisinin dâhil edilmesinin sağladığı avantajı vurgulamakta ve önerilen yaklaşımın

ters Nadarajah–Haghighi dağılımı için daha etkili bir çıkarım çerçevesi sunduğunu göstermektedir. Bu çalışma, söz konusu dağılım altında parametre tahmini için örnekleme tasarımlarının kapsamlı bir karşılaştırmasını sunması ve medyan sıralı kümeleme örneklemesiyle elde edilen pratik kazanımları ortaya koyması bakımından literatüre katkı sağlamaktadır.

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Anahtar Kelimeler: Medyan sıralı kümeleme örneklemesi, basit rastgele örnekleme, en çok olabilirlik tahmin edicisi, ters Nadarajah-Haghighi dağılımı

Introduction

The distribution most frequently used in life testing and reliability studies was the exponential distribution with a constant failure rate. However, researchers have turned to different exponential distribution generalizations that show non-monotonic failure rate behaviors in their model selection. One of the most important of these generalizations is the Nadarajah-Haghighi (NH) distribution proposed by Nadaraj and Haghighi [1]. The NH distribution is characterized by the following cumulative distribution function (cdf) and probability density function (pdf):

$$F(x) = 1 - exp(1 - (1 + \lambda x)^{\alpha}) \quad (1)$$

$$f(x) = \alpha\lambda(1 + \lambda x)^{(\alpha-1)} exp(1 - (1 + \lambda x)^{\alpha}) \quad (2)$$

where α and λ are the shape and scale parameters. In contrast, inverted distributions have been introduced in the literature by applying the inverse transformation. A few notable inverted models are: inverted Rayleigh [2], inverted Weibull [3], inverted exponential [4], inverted Lindley [5], inverted gamma [6] and inverted power Lindley [7]. Most recently, [8] introduced the inverted Nadaraj-Haghighi (INH) distribution, which is a highly flexible distribution with a decreasing and unimodal density as well as decreasing and upside-down bathtub-shaped hazard rates. In their article, they focused on the Bayesian estimation and mathematical properties of the INH distribution. Several researchers have examined the distribution of INH and introduced new extensions by incorporating additional parameters, see for instance, Raffiq et al. [9] proposed Marshall-Olkin INH, Toumaj et al. [10] proposed transmuted INH, Ahsan-ul-Haq et al. [11] power series INH. Presented below are the cdf and pdf of the INH distribution:

$$F(x) = exp(1 - (1 + \lambda x^{-1})^{\alpha}) \quad (3)$$

$$f(x) = \alpha\lambda x^{-2}(1 + \lambda x^{-1})^{\alpha-1} exp(1 - (1 + \lambda x^{-1})^{\alpha}) \quad (4)$$

where α and λ denote the shape and scale parameters. A number of researchers have conducted extensive studies on the statistical inference of unknown parameters associated with the INH distribution. For example, Elshahhat and Rastogi [12] considered unknown parameters of the INH distribution using ML and maximum product of spacings estimators, Elshahhat et al. [13] tackled the challenge of estimating these parameters and their reliability functions under generalized Type-II progressive hybrid censoring, employing both ML and Bayesian estimation methods, Abo-Kasem et al. [14] estimated unknown parameters of the INH using Bayesian methods, Abushal and Al-Zaydi [15] investigated the inference of unknown parameters for the INHD through a generalized Type-II hybrid censoring approach. Muttlak [16] introduced Median Ranked Set Sampling (MRSS) an alternative to Ranked Set Sampling (RSS) for estimating the population mean, aiming to reduce errors and improve efficiency. The process is detailed in the following steps:

1. Randomly select m^2 units from the population.
2. Divide this m^2 units into m sets, each containing m units, and rank the units within each set.
3. From each set, select one unit according to the value of m:

- If m is odd, select the unit with rank $(m+1)/2$ from each set.
- If m is even, select the unit with rank $m/2$ from the first $m/2$ sets and the unit with rank $(m+2)/2$ from the remaining $m/2$ sets.

4. Repeat Steps 1–3 for h cycles to obtain a final MRSS sample of size $n = hm$.

The procedure for one cycle and case of m=5 and m=6 can be shown in Figures 1-2.

$X_{1[1,5]1}$	$X_{1[2,5]1}$	$X_{1[3,5]1}$	$X_{1[4,5]1}$	$X_{1[5,5]1}$
$X_{2[1,5]1}$	$X_{2[2,5]1}$	$X_{2[3,5]1}$	$X_{2[4,5]1}$	$X_{2[5,5]1}$
$X_{3[1,5]1}$	$X_{3[2,5]1}$	$X_{3[3,5]1}$	$X_{3[4,5]1}$	$X_{3[5,5]1}$
$X_{4[1,5]1}$	$X_{4[2,5]1}$	$X_{4[3,5]1}$	$X_{4[4,5]1}$	$X_{4[5,5]1}$
$X_{5[1,5]1}$	$X_{5[2,5]1}$	$X_{5[3,5]1}$	$X_{5[4,5]1}$	$X_{5[5,5]1}$

Figure 1. *MRSS design in case of m=5.*

$X_{1[1,6]1}$	$X_{1[2,6]1}$	$X_{1[3,6]1}$	$X_{1[4,6]1}$	$X_{1[5,6]1}$	$X_{1[5,6]1}$
$X_{2[1,6]1}$	$X_{2[2,6]1}$	$X_{2[3,6]1}$	$X_{2[4,6]1}$	$X_{2[5,6]1}$	$X_{2[5,6]1}$
$X_{3[1,6]1}$	$X_{3[2,6]1}$	$X_{3[3,6]1}$	$X_{3[4,6]1}$	$X_{3[5,6]1}$	$X_{3[5,6]1}$
$X_{4[1,6]1}$	$X_{4[2,6]1}$	$X_{4[3,6]1}$	$X_{4[4,6]1}$	$X_{4[5,6]1}$	$X_{4[5,6]1}$
$X_{5[1,6]1}$	$X_{5[2,6]1}$	$X_{5[3,6]1}$	$X_{5[4,6]1}$	$X_{5[5,6]1}$	$X_{5[5,6]1}$
$X_{6[1,6]1}$	$X_{6[2,6]1}$	$X_{6[3,6]1}$	$X_{6[4,6]1}$	$X_{6[6,6]1}$	$X_{6[6,6]1}$

Figure 2. *MRSS design in case of m=6.*

Several authors have shown interest in using MRSS for parameter estimation in different distribution. Some recently used alternative distributions are Cauchy by [17], Lomax by [18], Pareto by [19], Weibull-Pareto by [20]-[21], generalized Rayleigh by [22], Rayleigh by [23], exponentiated exponential by [24], log-logistic by [25]. Building on this line of research, the main contributions of this study are summarized as follows; this study derives the maximum likelihood estimators of the parameters of the inverted Nadarajah–Haghighi distribution under both simple random sampling and median ranked set sampling schemes, providing explicit likelihood formulations for each design. A comprehensive comparison between simple random sampling and median ranked set sampling is conducted within the same distributional framework, allowing the efficiency gains of incorporating ranking information to be clearly quantified. An extensive Monte Carlo simulation study is performed to evaluate the estimators in terms of bias and mean squared error across a wide range of sample sizes and parameter values, offering a robust assessment of their finite-sample behavior. The results demonstrate that median ranked set sampling consistently yields more efficient and precise parameter estimates than simple random sampling, highlighting its practical advantage for inference on the inverted Nadarajah–Haghighi distribution. To the best of the authors' knowledge, this study represents one of the first systematic investigations of parameter estimation for the inverted Nadarajah–Haghighi distribution based on

median ranked set sampling, thereby contributing new insights to the ranked set sampling and lifetime distribution literature.

Materials and Methods

Estimation of the Inverted Nadarajah Haghighi Distribution Parameters

In this section, the MLE of the parameters α and λ for INH distribution based on SRS and MRSS is presented.

Using RSS

Let $X_1, X_2, \dots, X_n$ denote a random sample of size n drawn from INH distribution. The corresponding likelihood function (L) is:

$$L(\alpha,\lambda;x) = \prod_{i=1}^{n} \alpha\lambda x_i^{-2}\left(1+\lambda x_i^{-1}\right)^{\alpha-1} \times exp\left(1-\left(1+\lambda x_i^{-1}\right)^{\alpha}\right) \quad (5)$$

and the log-likelihood function (l) is

$$l(\alpha,\lambda;x) = nlog(\alpha\lambda) - 2\sum_{i=1}^{n} log(x_i) + (\alpha-1)\sum_{i=1}^{n} log\left(1+\lambda x_i^{-1}\right) + \sum_{i=1}^{n} 1-\left(1+\lambda x_i^{-1}\right)^{\alpha} \quad (6)$$

Taking the derivative of the log-likelihood function with respect to α and λ gives the following results:

$$\frac{\partial l(\alpha,\lambda;x)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log\left(1+\lambda x_i^{-1}\right) - \sum_{i=1}^{n}\left(1+\lambda x_i^{-1}\right)^{\alpha} log\left(1+\lambda x_i^{-1}\right) \quad (7)$$

$$\frac{\partial l(\alpha,\lambda;x)}{\partial \lambda} = \frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n} \frac{x_i^{-1}}{\left(1+\lambda x_i^{-1}\right)} - \alpha\sum_{i=1}^{n} x_i^{-1}\left(1+\lambda x_i^{-1}\right)^{\alpha-1} \quad (8)$$

Using MRSS

Odd Set Sizes

Suppose $X_1, X_2, \dots, X_n$ be a random sample of size n selected from INH distribution $f(\alpha,\lambda;x)$. Let $W = \{W_{ij}, i = 1,\dots,m, j = 1,\dots,r\}$ be a MRSS; that is

$$W_{ij} = \left\{ X_{\frac{m+1}{2}} \quad if\ m\ is\ odd, i = 1,\dots,m\ \&\ j = 1,\dots,r \right.$$

The pdf of W_{ij} is

$$h_{W_{ij}}(w) = \left\{ h_{\frac{m+1}{2}}\left(W_{ij}\right) = f_{X_{\frac{m+1}{2}}}(w) \quad if\ m\ is\ odd, i = 1,\dots,m\ \&\ j = 1,\dots,r. \right.$$

For odd set size m, the L of MRSS is

$$L(\alpha,\lambda;\mathrm{w}) = \prod_{j=1}^{r}\prod_{i=1}^{m} h\left(w_{ij\left(\frac{m+1}{2}\right)};\ \alpha,\lambda\right) = (C_1)^{mr}\alpha^{mr}\lambda^{-mr}$$

$$\times \prod_{j=1}^{r}\prod_{i=1}^{m}\left(w_{ij\left(\frac{m+1}{2}\right)}\right)^{-2}\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha-1} exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)$$

$$\times\left(exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)\right)^{\frac{m-1}{2}} \times\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)\right)^{\frac{m-1}{2}} \quad (9)$$

where $C_1 = \frac{m!}{\left(\frac{m-1}{2}\right)!^2}$. The l is

$$l(\alpha,\lambda;w) = mr\ln(C_1) + mr\ln(\alpha) + mr\ln(\lambda) - 2\sum_{j=1}^{r}\sum_{i=1}^{m} log\left(w_{ij\left(\frac{m+1}{2}\right)}\right)$$

$$-2\sum_{j=1}^{r}\sum_{i=1}^{m} log\left(w_{ij\left(\frac{m+1}{2}\right)}\right)+\sum_{j=1}^{r}\sum_{i=1}^{m}\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)$$

$$+\sum_{j=1}^{r}\sum_{i=1}^{m}\left(\frac{m-1}{2}\right)\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)$$

$$+\sum_{j=1}^{r}\sum_{i=1}^{m}\left(\frac{m-1}{2}\right) log\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)\right) \quad (10)$$

Taking the derivative $l(\alpha,\lambda;w)$ with respect to α and λ, gives

$$\frac{\partial l(\alpha,\lambda;w)}{\partial\alpha}=\frac{mr}{\alpha}+\sum_{j=1}^{r}\sum_{i=1}^{m}\log\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)-\sum_{j=1}^{r}\sum_{i=1}^{m}\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}$$

$$\times\log\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)-\sum_{j=1}^{r}\sum_{i=1}^{m}\left(\frac{m-1}{2}\right)\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\times\log\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)$$

$$+\sum_{j=1}^{r}\sum_{i=1}^{m}\frac{\left(\frac{m-1}{2}\right)\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}\times\frac{\log\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}$$

$$\times\frac{exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}=0 \quad (11)$$

$$\frac{\partial l(\alpha,\lambda;w)}{\partial\lambda}=\frac{mr}{\lambda}+(\alpha-1)\sum_{j=1}^{r}\sum_{i=1}^{m}\frac{w_{ij\left(\frac{m+1}{2}\right)}^{-1}}{\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)}-\alpha\sum_{j=1}^{r}\sum_{i=1}^{m}w_{ij\left(\frac{m+1}{2}\right)}^{-1}\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha-1}$$

$$-\alpha\sum_{j=1}^{r}\sum_{i=1}^{m}\left(\frac{m-1}{2}\right)w_{ij\left(\frac{m+1}{2}\right)}^{-1}\times\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha-1}$$

$$+\alpha\sum_{j=1}^{r}\sum_{i=1}^{m}\frac{\left(\frac{m-1}{2}\right)w_{ij\left(\frac{m+1}{2}\right)}^{-1}\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha-1}}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}\times\frac{exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+1}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}=0 \quad (12)$$

Even Set Sizes

Suppose $X_1, X_2, \dots, X_n$ be a random sample of size *n* selected from INH distribution $f(\alpha,\lambda;x)$. Let $W=\{W_{ij}, i=1,\dots,m, j=1,\dots,r\}$ be a MRSS; that is

$$W_{ij}=\begin{cases} X_{\frac{m}{2}},\ if\ m\ is\ even, i=1,\dots,\frac{m}{2},\ j=1,\dots,r \\ X_{\frac{m+2}{2}},\ if\ m\ is\ even, i=\frac{m+2}{2},\dots,m,\ j=1,\dots,r \end{cases}$$

The pdf of W_{ij} is

$$h_{W_{ij}}(w)=h_{\frac{m}{2}}(W_{ij})=f_{X_{\frac{m}{2}}}(w),$$

$$h_{W_{ij}}(w)=h_{\frac{m}{2}}(W_{ij})=f_{X_{\frac{m}{2}}}(w),$$

$$h_{\frac{m+2}{2}}(W_{ij})=f_{X_{\frac{m+2}{2}}}(w)$$

$if\ m\ is\ even, i = \frac{m+2}{2}, \dots, m,\ j = 1, \dots, r$

For even set size m, the L of MRSS is

$$L(\alpha,\lambda;\mathrm{w}) = \prod_{j=1}^{r}\prod_{i=1}^{m/2} h\left(w_{ij\left(\frac{m}{2}\right)};\ \alpha,\lambda\right)\prod_{j=1}^{r}\prod_{i=(m+2)/2}^{m} h\left(w_{ij\left(\frac{m+2}{2}\right)};\ \alpha,\lambda\right)$$

$$= (C_2)^{mr}\alpha^{mr}\lambda^{-mr}\prod_{j=1}^{r}\prod_{i=1}^{m/2}\left(w_{ij\left(\frac{m}{2}\right)}\right)^{-2}\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha-1} exp\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)$$

$$\times\left(exp\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)\right)^{\frac{m-1}{2}}\times\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)\right)^{\frac{m-1}{2}}$$

$$\times\prod_{j=1}^{r}\prod_{i=(m+2)/2}^{m}\left(w_{ij\left(\frac{m+2}{2}\right)}\right)^{-2}\times\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha-1}\times exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha}\right)$$

$$\times\left(exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha}\right)\right)^{\frac{m-1}{2}}\times\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha}\right)\right)^{\frac{m-1}{2}} \quad (13)$$

where $C_2 = \left(\frac{m!}{\left(\frac{m}{2}\right)!\left(\frac{m}{2}-1\right)!}\right)^2$. The l is

$$l(\alpha,\lambda;w) = mr\ln(C_2) + mr\ln(\alpha) + mr\ln(\lambda) - 2\sum_{j=1}^{r}\sum_{i=1}^{m/2} log\left(w_{ij\left(\frac{m}{2}\right)}\right)$$

$$+(\alpha-1)\sum_{j=1}^{r}\sum_{i=1}^{m/2} log\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right) + (\alpha-1)\sum_{j=1}^{r}\sum_{i=1}^{m/2} log\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)$$

$$+\sum_{j=1}^{r}\sum_{i=1}^{m/2}\left(\frac{m-1}{2}\right)\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)+\sum_{j=1}^{r}\sum_{i=1}^{m/2}\left(\frac{m-1}{2}\right)\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)$$

$$-2\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m} log\left(w_{ij\left(\frac{m+2}{2}\right)}\right) + (\alpha-1)\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m} log\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)$$

$$+\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m}\left(1-\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha}\right)+\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m}\left(\frac{m-1}{2}\right)\left(1-\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha}\right)$$

$$+\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m}\left(\frac{m-1}{2}\right)\left(1-\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha}\right) \quad (14)$$

Taking the derivative $l(\alpha,\lambda;w)$ with respect to α and λ, gives

$$\frac{\partial l(\alpha,\lambda;w)}{\partial\alpha} = \frac{mr}{\alpha} + \sum_{j=1}^{r}\sum_{i=1}^{m/2}\log\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right) - \sum_{j=1}^{r}\sum_{i=1}^{m/2}\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}$$

$$\times\log\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right) - \sum_{j=1}^{r}\sum_{i=1}^{m/2}\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\times\log\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)$$

$$\times\log\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right) + \sum_{j=1}^{r}\sum_{i=1}^{m/2}\frac{\left(\frac{m-1}{2}\right)\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}\times\frac{\log\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}$$

$$\times\frac{exp\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)\right)} = 0. \quad (15)$$

$$\frac{\partial l(\alpha,\lambda;w)}{\partial \lambda}=\frac{mr}{\lambda}+(\alpha-1)\sum_{j=1}^{r}\sum_{i=1}^{m/2}\frac{w_{ij\left(\frac{m}{2}\right)}^{-1}}{\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)}-\alpha\sum_{j=1}^{r}\sum_{i=1}^{m/2}w_{ij\left(\frac{m}{2}\right)}^{-1}\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha-1}$$

$$-\alpha\sum_{j=1}^{r}\sum_{i=1}^{m/2}w_{ij\left(\frac{m}{2}\right)}^{-1}\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha-1}+\alpha\sum_{j=1}^{r}\sum_{i=1}^{m/2}\frac{\left(\frac{m-1}{2}\right)w_{ij\left(\frac{m}{2}\right)}^{-1}\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha-1}}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}$$

$$\times\frac{exp\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}+(\alpha-1)\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m}\frac{w_{ij\left(\frac{m+2}{2}\right)}^{-1}}{\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)}$$

$$-\alpha\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m}w_{ij\left(\frac{m+2}{2}\right)}^{-1}\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha-1}$$

$$-\alpha\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m}\left(\frac{m-1}{2}\right)w_{ij\left(\frac{m+2}{2}\right)}^{-1}\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha-1}$$

$$+\alpha\sum_{j=1}^{r}\sum_{i=(m+2)/2}^{m}\frac{\left(\frac{m-1}{2}\right)w_{ij\left(\frac{m+2}{2}\right)}^{-1}}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}\times\frac{\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha-1}}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}$$

$$\times\frac{exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha}\right)}{\left(1-exp\left(1-\left(1+\lambda w_{ij\left(\frac{m+2}{2}\right)}^{-1}\right)^{\alpha}\right)\right)}. \tag{16}$$

Due to the absence of a closed-form solution for the MLEs under both even and odd set sizes m, equations (11), (12), (15), and (16) are resolved using numerical methods.

Results and Discussion

Simulation Results

To explore the properties of the MLE for the parameters of the INH distribution, a simulation study was conducted. A Monte Carlo simulation was performed using different sample sizes, $m = (2,3,4)$ and $n = 20,50,100$, and various parameter values, including scale parameters $\lambda = 1$ and shape parameter $\alpha = 0.4,1.1,2.0,3.0$. For both SRS and MRSS, 10,000 Monte Carlo replicates were generated by drawing m random samples from the INH distribution using the quantile function given below:

$$x=\lambda\left[(1-\ln(u))^{1/\alpha}-1\right]^{-1} \tag{17}$$

To assess the performance of each sample design in estimation, Bias and MSE measures are taken into account. The formulas are given below:

$$Bias\left(\hat{\theta}_i,\theta\right)=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i-\theta\right) \tag{18}$$

$$MSE\left(\hat{\theta}_i,\theta\right)=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i-\theta\right)^2 \tag{19}$$

The results are reported in Tables 1-3.

Table 1. *Estimated MSE and bias values for the parameter, α=0.1.*

λ	N	*m;r*	MRSS				SRS			
			MSE		Bias		MSE		Bias	
			$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
0.1	24	3;8	0.0004	0.0395	0.0067	0.0366	0.0014	0.0503	0.0070	0.0460
		4;6	0.0004	0.0434	0.0064	0.0396				
		6;4	0.0005	0.0442	0.0068	0.0404				
	36	3;12	0.0002	0.0156	0.0044	0.0231	0.0009	0.0394	0.0054	0.0308
		4;9	0.0002	0.0167	0.0043	0.0252				
		6;6	0.0002	0.0139	0.0041	0.0233				
	48	3;16	0.0002	0.0077	0.0030	0.0145	0.0005	0.0135	0.0043	0.0196
		4;12	0.0002	0.0093	0.0030	0.0177				
		6;8	0.0002	0.0091	0.0029	0.0173				
0.5	24	3;8	0.0004	0.7863	0.0068	0.1711	0.0013	1.8161	0.0086	0.4636
		4;6	0.0004	1.1085	0.0061	0.2136				
		6;4	0.0005	1.4513	0.0077	0.3454				
	36	3;12	0.0002	0.3670	0.0041	0.1110	0.0009	0.5124	0.0048	0.2015
		4;9	0.0002	0.3833	0.0039	0.1207				
		6;6	0.0002	0.3732	0.0042	0.1183				
	48	3;16	0.0002	0.2095	0.0032	0.0775	0.0006	0.3370	0.0037	0.1449
		4;12	0.0002	0.2215	0.0029	0.0839				
		6;8	0.0002	0.2539	0.0031	0.0955				
1.0	24	3;8	0.0004	3.5039	0.0066	0.3480	0.0014	4.4551	0.0069	0.5374
		4;6	0.0004	4.3303	0.0063	0.4338				
		6;4	0.0004	4.2342	0.0065	0.4186				
	36	3;12	0.0002	1.2721	0.0043	0.1982	0.0009	1.9404	0.0049	0.3113
		4;9	0.0002	1.5908	0.0043	0.2251				
		6;6	0.0002	1.7655	0.0040	0.2636				
	48	3;16	0.0001	0.8983	0.0031	0.1562	0.0006	1.1883	0.0036	0.2556
		4;12	0.0002	0.9328	0.0028	0.1863				
		6;8	0.0002	1.0880	0.0028	0.2035				
2.0	24	3;8	0.0004	6.2238	0.0066	0.7527	0.0014	14.4055	0.0078	0.9156
		4;6	0.0004	7.9961	0.0061	0.8595				
		6;4	0.0004	5.9991	0.0068	0.7371				
	36	3;12	0.0002	2.6525	0.0041	0.4571	0.0009	5.2900	0.0051	0.5266
		4;9	0.0002	3.1615	0.0043	0.4563				
		6;6	0.0002	3.5744	0.0042	0.4680				
	48	3;16	0.0001	1.3320	0.0030	0.3032	0.0005	3.5103	0.0035	0.3970
		4;12	0.0001	1.8543	0.0028	0.3596				
		6;8	0.0001	1.6586	0.0031	0.3476				

Table 2. *Estimated MSE and bias values for the parameter, α=0.3.*

λ	N	*m;r*	MRSS				SRS			
			MSE		Bias		MSE		Bias	
			$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
0.1	24	3;8	0.0245	0.0114	0.0454	0.0125	1.9530	0.0187	0.0802	0.0142
		4;6	0.0576	0.0077	0.0487	0.0096				
		6;4	1.7653	0.0090	0.0735	0.0121				
	36	3;12	0.0062	0.0049	0.0260	0.0071	0.0089	0.0074	0.0281	0.0083
		4;9	0.0067	0.0046	0.0257	0.0069				
		6;6	0.0071	0.0046	0.0259	0.0077				
	48	3;16	0.0037	0.0032	0.0178	0.0057	0.0056	0.0041	0.0190	0.0070
		4;12	0.0040	0.0028	0.0176	0.0047				
		6;8	0.0042	0.0028	0.0178	0.0044				
0.5	24	3;8	0.0183	0.2288	0.0456	0.0498	0.1206	0.3213	0.0540	0.0823
		4;6	0.0309	0.2589	0.0434	0.0752				
		6;4	0.1095	0.2439	0.0508	0.0633				
	36	3;12	0.0066	0.1261	0.0257	0.0391	0.0194	0.1456	0.0380	0.0586
		4;9	0.0069	0.1139	0.0246	0.0368				
		6;6	0.0167	0.1331	0.0328	0.0463				
	48	3;16	0.0037	0.0776	0.0183	0.0239	0.0067	0.0925	0.0213	0.0370
		4;12	0.0041	0.0734	0.0174	0.0224				
		6;8	0.0042	0.0778	0.0182	0.0267				
1.0	24	3;8	0.5110	0.9904	0.0545	0.1140	0.7941	1.3891	0.0760	0.1718
		4;6	0.7212	0.8325	0.0690	0.1123				
		6;4	0.3061	1.1023	0.0517	0.1549				
	36	3;12	0.0058	0.5024	0.0254	0.0757	0.0084	0.5720	0.0294	0.0791
		4;9	0.0075	0.4256	0.0268	0.0652				
		6;6	0.0071	0.4582	0.0257	0.0738				
	48	3;16	0.0039	0.3260	0.0188	0.0533	0.0056	0.4128	0.0196	0.0551
		4;12	0.0040	0.2843	0.0181	0.0478				
		6;8	0.0044	0.2808	0.0189	0.0430				
2.0	24	3;8	0.4641	3.4994	0.0557	0.2054	0.7597	3.9510	0.0644	0.2587
		4;6	0.0263	3.3500	0.0434	0.2166				
		6;4	0.6187	3.6291	0.0614	0.2383				
	36	3;12	0.0066	1.8388	0.0266	0.1311	0.0089	2.1086	0.0314	0.2017
		4;9	0.0072	1.7392	0.0247	0.1392				
		6;6	0.0071	1.9268	0.0242	0.1791				
	48	3;16	0.0037	1.2979	0.0184	0.0974	0.0048	1.3567	0.0208	0.1181
		4;12	0.0039	1.1471	0.0179	0.0893				
		6;8	0.0032	1.2560	0.0172	0.1098				

Table 3. *Estimated MSE and bias values for the parameter, α=0.5.*

λ	N	*m;r*	MRSS				SRS			
			MSE		Bias		MSE		Bias	
			$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
0.1	24	3;8	5.4095	0.0085	0.3919	0.0072	11.5721	0.0113	0.6167	0.0087
		4;6	6.5074	0.0072	0.5026	0.0061				
		6;4	5.6299	0.0064	0.4604	0.0053				
	36	3;12	1.6099	0.0043	0.1084	0.0040	2.8212	0.0062	0.1661	0.0057
		4;9	1.7136	0.0037	0.1162	0.0017				
		6;6	1.9991	0.0039	0.1311	0.0021				
	48	3;16	0.0301	0.0028	0.0558	0.0029	0.4222	0.0039	0.0647	0.0034
		4;12	0.0342	0.0025	0.0608	0.0015				
		6;8	0.0374	0.0026	0.0571	0.0021				
0.5	24	3;8	6.9819	0.2114	0.5143	0.0398	13.9020	0.2461	0.6694	0.0469
		4;6	7.4381	0.1863	0.5365	0.0268				
		6;4	8.5978	0.1792	0.5555	0.0282				
	36	3;12	1.4395	0.1039	0.0977	0.0168	3.0124	0.1242	0.1397	0.0194
		4;9	1.5960	0.0883	0.1100	0.0098				
		6;6	1.3447	0.0967	0.0911	0.0166				
	48	3;16	0.5355	0.0790	0.0596	0.0130	0.5939	0.0842	0.0682	0.0152
		4;12	0.3882	0.0615	0.0568	0.0073				
		6;8	0.5824	0.0607	0.0624	0.0061				
1.0	24	3;8	9.8952	0.7096	0.4951	0.0406	18.2183	0.8434	0.5397	0.0669
		4;6	7.7430	0.6813	0.4749	0.0381				
		6;4	8.6676	0.6876	0.4861	0.0415				
	36	3;12	5.1168	0.4246	0.1398	0.0428	3.4271	0.5271	0.1588	0.0524
		4;9	2.3448	0.3705	0.1068	0.0305				
		6;6	3.3204	0.3600	0.1244	0.0249				
	48	3;16	1.0178	0.2909	0.0701	0.0249	1.3397	0.3742	0.0709	0.0397
		4;12	0.0394	0.2436	0.0548	0.0165				
		6;8	0.0320	0.2037	0.0575	0.0091				
2.0	24	3;8	8.6522	3.4428	0.4396	0.1521	15.8456	3.9251	0.6264	0.2064
		4;6	7.0694	2.6793	0.4142	0.0793				
		6;4	9.5870	2.9087	0.5488	0.0864				
	36	3;12	1.7235	0.9889	0.1064	0.0570	2.7133	1.5214	0.1374	0.0747
		4;9	1.4804	0.8176	0.1019	0.0360				
		6;6	1.5803	1.2029	0.1180	0.0660				
	48	3;16	1.2062	0.2062	0.0593	0.0392	1.5396	0.3225	0.0674	0.0435
		4;12	0.9676	0.0330	0.0562	0.0203				
		6;8	0.9820	0.0357	0.0552	0.0284				

Based on the tables the following findings can be inferred; the MSE associated with the parameters α and λ decreases with an increase in sample size. When the estimators are analyzed with respect to bias, MRSS is found to have lower bias values than SRS in every case. In a similar way, an analysis of the estimators in terms of MSE shows that the MSE values under MRSS are consistently lower than those under SRS across all scenarios. Under the MRSS design, when $\alpha = 0.1$, the α parameter demonstrates lower bias and MSE values than the λ parameter, but when $\alpha = 0.5$, the λ parameter presents lower bias and MSE values. For $\alpha = 0.3$ and $\lambda = 0.1$, the λ parameter exhibits lower bias and MSE values than the α parameter, while in other scenarios, α has lower values, as indicated in the Table 2. In the SRS design, with $\alpha = 0.1$, the λ parameter shows lower bias and MSE values than lambda, while at $\alpha = 0.5$, λ has lower bias and MSE values. Similar to MRSS, when $\alpha = 0.3$ and $\lambda = 0.1$, λ demonstrates lower values, whereas α shows lower values in other cases. The results obtained from the simulation study have several important implications for statistical inference based on the INH distribution. The

consistent reduction in bias and MSE under the MRSS design indicates that incorporating ranking information can substantially improve estimation accuracy without increasing the number of measured units. This finding is particularly relevant in practical applications where precise measurements are costly or time-consuming, such as reliability analysis and lifetime data modeling. However, the conclusions of this study should be interpreted in light of certain limitations. The simulation results are restricted to the specific parameter configurations and sample sizes considered, and the ranking process is assumed to be performed without error. In practice, imperfect ranking may affect estimator performance. Future research may extend the proposed framework by investigating the robustness of the estimators under ranking errors, exploring alternative ranked set sampling schemes, and considering censored or real-life data applications to further assess the practical utility of the proposed approach.

Conclusion

The INH distribution has been recognized as a flexible model for lifetime data exhibiting non-monotonic hazard rate structures. In this study, parameter estimation for this distribution was investigated under SRS and MRSS schemes using the maximum likelihood approach. Explicit likelihood formulations were derived for both sampling designs, and the resulting estimators were systematically compared through an extensive Monte Carlo simulation study. The simulation results clearly demonstrate that the MRSS yields more accurate and efficient parameter estimates than SRS in terms of bias and MSE, without requiring additional measured observations. This improvement highlights the practical benefit of incorporating ranking information into the sampling process, particularly in situations where direct measurements are costly or time-consuming. As a result, the proposed methodology provides a reliable and efficient inferential framework for the INH distribution. The findings of this study suggest that the proposed estimation strategy can be effectively applied in various fields where lifetime and reliability data arise, including engineering reliability, survival analysis, and biological and environmental studies. In addition to the maximum likelihood framework considered in this work, future research may extend the MRSS based inference for the INH distribution to Bayesian estimation approaches. Further studies may also explore the performance of alternative ranking based sampling schemes, such as stratified RSS, within the same distributional setting. Moreover, applying the proposed estimators to real data sets would provide additional insight into their practical performance. Overall, this study contributes to the existing literature by demonstrating the practical advantages of MRSS for parameter estimation and by offering a clear methodological reference for future research in ranked set sampling based inference.

Declarations

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Artificial Intelligence (AI): The author declare that AI tools (such as ChatGPT and Gemini) were used solely for language correction, translation, and readability enhancement during the preparation of this article. The author assume full responsibility for the originality, accuracy, and integrity of the entire text, including all AI-assisted language outputs.

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